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A logistics cost function with explicit transport costs

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ABSTRACT

With a view to construct a new framework to assess the benefits of freight transport improvements, an Economic Order Quantity model with uncertain lead time demand is equipped with detailed transport costs. The problem is to minimise total logistics cost by choosing shipment size, vehicle size and reorder point subject to constraints on vehicle size and annual transport capacity. An analytical solution in all variables except the reorder point is derived, reducing the cost minimisation problem to a well-behaved problem in one dimension only.

Different parts of transport costs influence the solution differently: Some act like ordering costs, some like holding costs and some have no influence on the solution. The solution exhibits economies of scale at all levels of optimal shipment size.

Examples with real data show that model calibration for an entire population of firms is feasible at the firm level, and that the model produces reasonable results.

1. Introduction

Improving the speed and reliability of freight transport is considered a major transport policy objective in most countries, but the economic appraisal of reliability improvements in particular is not very well advanced. There are problems not just with assessing the value to firms and society of a certain reliability improvement, but also with assessing the size of the improvement itself.

This paper formulates an Economic Order Quantity (EOQ) model of the simplest possible supply chain where goods of a certain kind are shipped and transported from a point of production, say, to a sales outlet. Demand is of course the number of units requested by consumers at the outlet per unit of time. We assume it to be produced by a stationary stochastic process. But even though each demand realisation is a random variable, by the law of large numbers we treat annual demand as constant. Lead time is the time from when a new shipment is ordered until commodities are available on shelf. It too is a stochastic variable, for instance because of unreliable transport, which makes demand during lead time even more uncertain.

The value to society of measures to reduce transport time and improve reliability can be computed as the total cost reduction (including externalities) of the optimal solution of the model. To go with our model, a model to assess the size of the reliability improvement itself must be devised. This part of the problem is not addressed here.

A main idea is that in contrast to passenger transport, firms will already have made provisions to insure against the consequences of delays and unexpected incidents, namely in the form of a safety stock designed to maintain an optimal level of service. What this level is should however not be decided in advance but be the outcome of minimisation of all relevant costs. A policy measure that reduces expected transport time or uncertainty in transport will make it possible for firms to reduce their safety stocks at the same level of service or to improve the level of service at the same level of safety stock. In addition, there will be transport cost savings as vehicles can carry out more jobs per day, and as reliability improvements may reduce the planned slack between jobs.

Starting with the seminal paper by Baumol and Vinod (1970), there is a fairly large literature where transport costs are added to an inventory theory model, either to model mode choice or to get a better grasp on the different elements of total logistics cost. The mode choice situation is considered by, among others, Das (1974), Constable and Whybark (1978), Langley (1981), Larson (1988), Combes (2009) and Lloret-Batlle and Combes (2012). Das (1975) and Tyworth and Zeng (1998) consider the impact of uncertain transport time on total costs, and Allen and Liu (1993) and Larson (1989) treat loss and damage costs, including the costs of quality control. Vernimmen and Witlox (2003) review the literature.

The official Swedish and Norwegian freight models are examples of modelling frameworks that incorporate explicit transport costs, choice of transport chain and shipment size as well as EOQ storage costs. However, parameters are based on aggregate data, and safety stocks and stock-out costs are not included (Hovi et al., 2018).

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The statistical distribution of uncertain demand during lead time is taken up in Hadley and Whitin (1963), Das (1976, 1983), Tyworth (1991, 1992), Namit and Chen (1999), Tyworth and Ganeshan (2000), Cobb (2013), Cobb et al. (2013), Cobb et al. (2015) and Wanke and Leiva (2015), among others. Vernimmen et al. (2008) review this strand of literature. Various indicators and decision criteria for fixing the shipment size and reorder point have been suggested in the literature. Tyworth (2018) shows how some of these give rise to paradoxes, and how the paradoxes may be avoided.

This paper contributes to the literature by a more detailed modelling of the transport and loading and unloading costs, and by recognising that, with costs increasing in vehicle size and upper and lower bounds on the vehicle size, the optimal shipment size may very well be smaller than the smallest vehicle, but it will never be optimal to use a vehicle with more capacity than strictly necessary. Overall, transport costs, often ignored in EOQ models, will be shown to be highly relevant when determining the optimal order quantity in our model. Furthermore, by using the optimised total cost function to assess policies and exogeneous changes instead of pre-set criteria or operational rules, the problems treated by Tyworth (2018) do not appear.

Most of the literature that explicitly considers transport costs in the context of logistics costs minimisation treats costs as they appear to the decision maker (usually a private firm). Thus, transport costs are the prices paid to the transport providers. Obviously, the true economic costs may diverge from these market prices if there is imperfect competition, external costs or economies of scale or scope – as there usually is in transport markets (Jansson, 1984). Key concerns driving the explicit and detailed modelling of transport costs that we are aiming for are therefore (1) to make it possible to substitute true economic costs for market prices for use in cost benefit analyses of freight transport improvements, (2) to study taxes and subsidies that may improve economic efficiency in the freight transport markets, and (3) to ensure that the model we derive below can be used as a sub model in a future freight transport model.

Section 2 provides an overview of the approach. In Section 3, we identify the elements of logistics cost. Section 4 formulates the logistics cost minimisation problem and finds the logistics cost function. Section 5 provides a test example using data from supply chains between the Netherlands and Norway. Section 6 concludes and points to future work.

2. Outline of the model

We consider a simple EOQ model where goods are supplied regularly from a source A to an outlet B (see Grubbström, 1995 for an historical overview of these models). Both demand and lead time are stochastic variables. Items are demanded one at a time, and orders do not cross. The inventory position is continuously reviewed, and a new order is placed as soon as it reaches a certain reorder point. Orders that cannot be met immediately are backordered. Transport may be door-to-door by truck or consist of two distribution stages by truck and a line haul by any mode.

The textbook cost minimisation problem in this case uses two policy variables, the shipment size and the reorder point, to minimise the sum of inventory costs, ordering costs, and stock-out costs. Transport costs are considered to be proportional to demand and thus irrelevant for the problem as long as expected demand per year is constant. However, a closer look at transport costs reveals that not all of them accrue per unit. The per unit transport costs are loading and unloading at terminals plus the cost of the line haul (assuming the line haul carrier offers a freight tariff by the unit). These costs may be left out of the cost minimisation problem. The picture is different for the stages where shipment size may affect the choice of vehicle size (vehicle capacity), i.e. distribution and door-to-door. If there are upper and lower bounds on vehicle size, the cost of using the smallest available vehicle will be a cost per shipment, or a part of the ordering cost, and highly relevant for the cost minimisation problem. The cost of choosing a larger vehicle size will also be relevant, with the implication that vehicle size must be included as a choice variable in the cost minimisation problem. This is shown in Section 3.1.

Now attention must be paid to uncertainty. Since lead time (including transport time as a part) and demand per hour (day, week) are both stochastic variables, demand during lead time (or lead time demand, LTD) is also a stochastic variable, as shown in Section 3.3. This being the case, there will usually be a non-zero probability of stock-outs, i.e. not being able to deliver from shelf. To allow the firm to adapt optimally against stock-outs, we assume that it may freely choose its reorder point. The reorder point is the inventory position that triggers the placement of a new order. A small reorder point makes for a high probability of stock-out during lead time, while a large reorder point makes for high average inventory costs. The difference between the reorder point and the expected LTD is the safety stock. Uncertainty about LTD means that both the inventory costs of the safety stock and the stock-out costs must be added to the logistics cost to be minimised. In this as in everything else except the transport costs, we follow standard inventory theory.

Obviously, the optimal safety stock and stock-out costs increase with uncertainty. Since transport time is a part of lead time, a policy measure that reduces transport time expectation and variance will reduce these elements of cost and thereby increase economic efficiency. Unreliable transport, on the other hand, gives rise to excessive safety stocks and/or high stock-out costs.

Apart from this, by the law of large numbers, the transport cost per shipment can still be based on expected transport time. Furthermore, assuming as we do that demand is generated by a stationary stochastic process, the number of shipments per year will admittedly vary from year to year, but over a number of years there will be an average expected annual cost of the operation. We assume that the objective of the decision maker in the model (which we define to be the shipper) is to minimise it. Its elements are the transport cost, non-transport ordering cost, the inventory holding cost – including the cost of holding a safety stock to guard against uncertain demand during lead time – and the stock-out costs.

For the model to reflect the true behaviour of the shipper, all costs must necessarily be as perceived by him, including taxes and charges. This does not prevent the model from being used for cost benefit analysis purposes or as elements of a national freight model. The transformation to social costs can be carried out by introducing a set of sectors, each with their own account, treating taxes, subsidies and charges as transfers between these sectors. External costs such as local and global transport emissions associated with the choices of the shipper must be added to an appropriate sector. Summing over sectors, the true social cost appears. Carriers (i.e. providers of transport services) and other subcontractors and suppliers to the supply chain must be assumed to earn zero profit, so that their true costs are perceived by the shipper.

3. The elements of logistics cost

We denote the firm in charge of the operation as the shipper, therefore the terms “firm” and “shipper” are used interchangeably and will always refer to the decision maker. The shipper's problem is to minimise all relevant logistics costs, including the relevant transport costs, with respect to shipment size, vehicle size and reorder point, subject to the constraints that shipment size cannot exceed the maximal vehicle size, vehicle size must be within a feasible range, and annual transport capacity must at least cover annual demand.

3.1. Transport costs

Transport is either door-to-door or combined transport. In the case of combined transport, we assume that the line haul freight tariff is a price P per unit of the commodity. We also assume that the frequency of the scheduled line haul transport service is sufficiently high not to constrain the choice of shipment size. With respect to the door-to-door transport or the distribution stages of the combined transport, we assume a perfect market. Besides zero profit to the carrier, we take this to mean that the firm in charge of the operation (the shipper) can always choose the most appropriate vehicle size C from a continuous interval $[C_{\min}, C_{\max}]$ and that outside the time when the vehicles are engaged in our supply chain, they will be fully employed in other engagements also yielding zero profit. (If there is a cost of moving the vehicles to the starting point of the operation, this can be included in the non-transport ordering costs.) It also means that the carrier can be counted upon to keep turn-around time and other unproductive time to a minimum. The vehicles employed in the door-to-door transport or the distribution stages of the combined transport are not supposed to pick up or carry other goods on the same tour. Thus empty backhaul is assumed. We define the following parameters:

x annual expected demand in units of the commodity
$\bar{t}$ expected one-way transport time in hours
u turn-around time and other unproductive time per round trip in hours
a one-way transport distance in kilometres
t_l combined expected loading and unloading time per unit in hours
k distance dependent costs per kilometre
i vehicle capital costs per hour
w the driver's wage cost per hour, assumed to apply both to $\bar{t}$ and t_l
w_l hourly cost of hiring loading and unloading capacity
η number of business hours per year
P the cost per unit of the line-haul

These parameters are all strictly positive and pertain to the distribution stages or door-to-door case. Where appropriate, subscripts on the parameters will be used to denote link or stage of the transport. It is assumed that wages and vehicle capital costs continue to accrue during turn-around time u. The model captures this. However, u might also act as a way for the carrier to keep an agreed schedule even in the face of incidents and delays, and this effect is not in the model.

The choice variables, also pertaining to the distribution stages or door-to-door-case, will be selected among the following:

Q shipment size in units of the commodity
C vehicle size (carrying capacity) in units of the commodity
Y maximum annual transport capacity in units of the commodity

Two linear relations will be assumed. Apart from the other assumptions and the values of exogenous variables and parameters, they constitute the empirical content of the model. First, we assume a linear relationship between vehicle size C and kilometre cost k:

$$k = k_0 + k_1 C \tag{1}$$

where k_0, k_1 are positive constants. Thus, we ignore the impact of speed on kilometre costs. Second, the relation between vehicle size and hourly vehicle capital cost is

$$i = i_0 + i_1 C \tag{2}$$

where i_0, i_1 are positive constants. For a given type of vehicles, both relations are thought to fit data well. For non-negative parameters, these two relations mean that provided the vehicle capacity can be utilised to the full, transport cost per tonne will be lower the larger the vehicle capacity. The kilometre cost k includes fuel, oil, tires and a part of maintenance and repair costs, while the vehicle capital cost i includes all vehicle costs that accrue per hour, day or year. All forms of taxes should normally be included.

Assuming no unnecessary trips are made, the number of trips per year will be x/Q. We use this to define the maximum annual transport capacity Y by:

$$Y \equiv C\frac{x}{Q} \tag{3}$$

Y is the annual quantity that *would* be shipped if the trucks were always filled to capacity. Given (1) and (2), it will never be optimal to operate larger vehicles than necessary. Thus $Y = x$ (and consequently $C = Q$) if $Q \in [C_{\min}, C_{\max}]$, and $Y = C_{\min}x/Q$ if $Q \leq C_{\min}$. Furthermore, $C_{\max}$ is treated as an upper bound on shipment size. This simplifies cost minimisation, since shipments never will have to be split between multiple vehicles.[1]

Let w be the cost per hour of crew, appropriately adjusted by the ratio of paid hours to hours "on the road". The hourly cost of engaging a vehicle will then be $w + i_0 + i_1 C$. The hourly cost of hiring loading and unloading capacity has been defined to be w_l. Since expected round trip time is $2\bar{t} + u + t_l Q$, the expected annual cost of transport K_T in the door-to-door case amounts to:

$$\begin{aligned} K_T &= (k_0 + k_1 C)2a\tfrac{x}{Q} + (w + i_0 + i_1 C)(2\bar{t} + u + t_l Q)\tfrac{x}{Q} + w_l t_l x = \\ &[2k_0 a + (w + i_0)(2\bar{t} + u)]\tfrac{x}{Q} + i_1 t_l QY + (w + i_0 + w_l)t_l \\ &x + [2k_1 a + i_1(2\bar{t} + u)]Y \end{aligned} \tag{4a}$$

In the first line of (4a), the first term is distance dependent costs per kilometre multiplied by distance and the number of trips; the second term is cost per hour multiplied by hours per trip and the number of trips; and the third term is the additional cost of loading and unloading operations. Equation (3) was used to eliminate C as a choice variable in the second line.

For the case of combined transport, let stage 1 be the first distribution stage, stage 2 be the line haul and stage 3 be the second distribution stage. The total cost of the second stage (including all transport cost elements, such as time costs, distance costs, and the cost of an extra loading and unloading) will be Px, which is to be added to the third term in the above expression. In the current article, Px is assumed to be given. However, if the present model were just an element of a larger freight transport model system, P should be determined as an equilibrium price in a larger transport market with economies of scale due to network externalities.

The shipment size and expected average annual number of shipments will be the same for both distribution stages, thus C and Y will be the same for both stages. The costs of each stage might however differ, as they depend on stage-specific a,$\bar{t}$ and u. Index the relevant variables of each stage by their stage number, let stage 1 be the only stage in the door-to-door case, and let δ be 1 in the combined transport case, 0 otherwise. The expected average annual transport costs are:

$$\begin{aligned} K_T &= \{2k_0(a_1 + \delta a_3) + (w + i_0)[2(\bar{t}_1 + \delta\bar{t}_3) + (u_1 + \delta u_3)]\}\tfrac{x}{Q} \\ &+ \{i_1 t_l\}QY + \{(w + i_0 + w_l)t_l + P\}x \\ &+ \{2k_1(a_1 + \delta a_3) + i_1[2(\bar{t}_1 + \delta\bar{t}_3) + (u_1 + \delta u_3)]\}Y \end{aligned} \tag{4b}$$

The first line of Equation (4b) shows that only the costs of the smallest possible vehicle (k_0 and i_0) accrue per shipment x/Q. The second and third line shows that the additional costs depend on annual demand and the vehicle size. Thus according to our model, to assume a fixed transport cost per shipment is only right if the vehicle size cannot be chosen, and to assume a fixed cost per tonne is only right if the costs of the line haul outweigh by far the cost of the distribution stages. Note that if $\delta = 0$, with some rearranging, Equation (4b) becomes equal to Equation (4a), so the latter is not needed anymore.

[1] This is also a realistic assumption in practice: While the transport costs per unit decrease in vehicle size (see equations (1) and (2)), section 3.2 and 3.4 will show that the inventory holding costs per unit increase in shipment size. Therefore, firms will never have any incentive to choose $Q > C_{\max}$ unless the reduction in ordering costs by far outweighs the increase in inventory costs.

3.2. Ordering costs and inventory holding costs, no uncertainty

We need the following prices, unit costs, and parameters:

b non-transport ordering costs (per shipment)
p free-on-board unit price of the commodity
H inventory holding cost per year and dollar of stationary inventory
J inventory holding cost per year and dollar of mobile inventory
ε inventory at the source as a share of inventory at the outlet

The non-transport ordering costs b includes any fixed cost associated with producing the commodities to be shipped (set-up costs) or with providing the transport service, as well as any cost of paperwork associated with the shipment. From the point of view of society, these costs at the source are relevant, and surely, they will also matter to the private decision maker. Since they occur per shipment, the expected annual ordering costs can be calculated as:

$$K_o = b\frac{x}{Q} \tag{5}$$

With no uncertainty, the inventory will decline at a constant rate from Q to zero (due to constant demand) and be restocked immediately after it is depleted (due to constant transport time). Thus, the average inventory at the outlet will be $\frac{1}{2}Q$. The annual cost of the stationary inventory with no uncertainty will be

$$\overline{K}_{SI} = \frac{1}{2}pH(1+\varepsilon)Q \tag{6}$$

This accounts for the inventory at the source as well as at the outlet. If production is instantaneous (import) and well-coordinated with shipments from the source, there need be no inventory at the source, but if the production rate equals the demand rate, average inventories at both points are equally large. Thus ε takes values between 0 and 1. H includes the cost of capital, variable warehousing costs and time-dependent depreciation (the case of stationary inventory costs under uncertainty is treated in section 3.4).

The mobile inventory will be proportional to annual demand and to average time in transport. The inventory holding cost per dollar tied up in transport, J, is not the same as H, since on one hand, there are no warehousing costs, and on the other, there might be a higher cost of depreciation and damage. Annual mobile inventory costs are equal to:

$$K_{MI} = pJ\eta^{-1}(\bar{t} + t_l Q)x \tag{7}$$

This relationship will still hold when uncertainty is taken into account, but in that case it reflects the expected annual mobile inventory cost.

3.3. The probability distribution of lead time demand

Next, we turn to uncertainty. When demand and lead time are stochastic, demand during lead time is also stochastic. We define the following parameters:

(μ_D,σ_D) Mean and standard deviation of demand per business hour ($\mu_D = x\eta^{-1}$)
(μ_T,σ_T) Mean and standard deviation of lead time in hours
(μ_L,σ_L) Mean and standard deviation of lead time demand (LTD) in units

Lead time, with transport time as a part, is the time from an order is placed until it is available at the outlet. It may be shown (Hadley and Whitin, 1963, exercise 3.12) that if demand per hour for all business hours are independently and identically distributed stochastic variables, LTD has the following expectation and variance:

$$\mu_L = \mu_D\mu_T \tag{8}$$

$$\sigma_L{}^2 = \mu_T\sigma_D^2 + \mu_D^2\sigma_T^2 \tag{9}$$

Other things equal, the longer the transport time, the longer the expected lead time. Thus Equation (9) shows that the longer the transport time, the more does the variance of demand contribute to LTD variance. Also, the mean level of demand will determine the impact of transport time variance. Clearly, LTD is important for the firm when it comes to the timing of a new order, since it determines the demand the firm can expect in the interval from a shipment is ordered until the commodities are available in shelf. Section 3.4 gives a formal treatment of this.

Note that regardless of LTD distribution, (8) and (9) are sufficient to determine its first two moments. However, the shape of the distribution will affect the logistics costs and therefore the firm's optimal policy. LTD has traditionally been assumed to be normally distributed for practical reasons (Tyworth, 1991). A commonly used argument for this assumption is that demand during each of the separate elements of lead time are independent stochastic variables, so that the central limit theorem applies. Vernimmen et al. (2008) claims that for this reason the normal distribution might be a good approximation for longer lead times "even in cases where the underlying distributions are highly asymmetric". Tyworth and O'Neil (1997) claim that assuming normally distributed LTD is robust for fast moving finished goods when the coefficient of variation $\sigma_L/\mu_L < 0.5$.

On the other hand, a recent body of research indicates that the normal distribution will often be unsuitable for real life scenarios. The most prominent alternative is the gamma distribution, according to two comprehensive literature surveys (Vernimmen et al., 2008; Cobb et al., 2013). There are three main reasons why this distribution is considered suitable (Burgin, 1975). First, it is only defined for non-negative values. Second, it can take a number of different shapes, ranging from an exponential distribution (the shape parameter equal to one), over a right skewed unimodal distribution, to an almost symmetric distribution similar to a normal (for high values of the shape parameter). Third, it is generally mathematically tractable under weak conditions in EOQ applications.

A third alternative is to determine the appropriate distributions for lead time and demand per time unit separately and construct a compound LTD distribution from these (see e.g. Bagchi et al., 1986; Tyworth, 1992). For aggregate economic analyses for which sensitive firm-specific data is difficult to obtain, creating compound distributions might be more convenient. This is because separate data sources will usually have to be used to determine the lead time and demand per time unit. However, since distribution functions is not the focus of this article, we stick to presenting two versions of the model; one in which LTD is normally distributed and one in which it follows a gamma distribution. For a detailed description of how these distributional assumptions affect the firm's minimisation problem, the reader is referred to Appendix A.

3.4. Safety stock and stock-out costs

This section will discuss how uncertainty affects the cost of the firm, and how the problem can be reformulated in order to let firms safeguard against the costs associated with this uncertainty. The following symbols will be relevant in this setting:

I Average physical (on hand) inventory excluding the mobile inventory
R Reorder point
E Expected number of backorders per year
B Expected number of backorders at any point in time
π The fixed unit cost per backorder
$\hat{\pi}$ The cost per year of a backorder

The firm trades off stock-out costs and inventory holding costs by fixing a reorder point R such that whenever the inventory position (on hand inventory plus shipments under way minus backorders) reaches R, a new shipment is ordered. We assume that the inventory is continually monitored and that items are demanded one at the time. This implies that a new shipment is ordered at the exact time the inventory position reaches the reorder point.

Stock-outs (not being able to deliver from shelf) are allowed to occur, but at a cost. We assume that stock-outs are backordered (delivered later) and allow backorders to have a cost per instance π. In addition, costs may depend on the time $\hat{\pi}$ until delivery can take place. Time-dependent stock-out costs are directly proportional to the time until backorders can be delivered multiplied by the number of stock-out instances. An example of this would be deliveries of intermediate products to a production line, where stock-outs imply setting the production on pause. From society's point of view, stock-out costs include the cost to the customer of having to wait, but for the firm they also mean lost goodwill.

Adding R as a policy instrument, we say that our firm is following a (Q,Y,R) policy. R might take on any value, including negative, which means that the firm collects backorders until they are so many as to make a new shipment worthwhile.

To be able to treat the case of stochastic lead time properly, we must assume that orders arrive in the same sequence in which they were placed. Let $h(X)$ be the probability density function of lead time demand X, such that $E(X) = \mu_L = \int_{\mathbb{R}} Xh(X)dX$. Note that the number of backorders during a replenishment cycle will be 0 if $X - R < 0$ and $X - R$ if $X - R \geq 0$. Thus, the expected number of backorders per year is

$$E(Q, R) = xQ^{-1}\int_{r}^{\infty} (X - R)h(X)dX = xQ^{-1}\left[\int_{r}^{\infty} Xh(X)dX - RH(R)\right] \quad (10)$$

i.e. the number of cycles multiplied by the expected backorders per cycle, where $H(X)$ is the complementary cumulative to $h(X)$ and xQ^{-1} is the number of replenishment cycles (throughout, lower-case x refers to annual demand, and must not be mistaken for the random variable LTD).

If we can determine the steady state probability density $g(\chi, Q, R)$ of the size of the inventory χ during a replenishment cycle, the expected number of backorders pending fulfilment at any point in time can be calculated as (Hadley and Whitin, 1963, pg. 194):

$$B(Q, R) = -\int_{\chi=-\infty}^{0} \chi g(\chi, Q, R)d\chi \quad (11)$$

The functional forms of $E(Q, R)$ and $B(Q, R)$ will depend on the distributional assumption on LTD. Normal and gamma distributed LDT are treated in detail in Appendix A. The average size of the inventory at the outlet can be written:

$$I = I(Q, R) = \frac{1}{2}Q + R - \mu_L + B(Q, R) \quad (12)$$

where the first term corresponds to the average inventory position without uncertainty and the second and third terms correspond to the additional safety stock implied by the reorder point. The fourth term corrects for the fact that while on hand inventory net of backorders can be negative, the inventory at the outlet – i.e. the number of units for which the stationary inventory cost applies – can never be smaller than zero. In other words, the stationary inventory cost can never be negative. Thus, when uncertainty is taken into account, the firm will have to pay stationary inventory costs for $R - \mu_L + B(Q, R)$ additional units at the outlet. This means that the stationary inventory costs under uncertainty can be written as:

$$K_{SI} = pH\left(\frac{1}{2}(1 + \varepsilon)Q + R - \mu_L + B(Q, R)\right) \quad (13)$$

When firms choose their optimal reorder point, the cost of holding this additional safety stock must be balanced against the expected stock-out costs, which will be:

$$K_{SO} = \pi E(Q, R) + \hat{\pi}B(Q, R) \quad (14)$$

Note that $B(Q,R)$ fills two roles in the firm's cost minimization problem. On one hand, backorders pending fulfilment are relevant to calculate the size of the inventory. On the other hand, $B(Q,R)$ can be used directly to calculate the backorder costs that are proportional to the time until delivery takes place.

4. Annual logistics costs

We are now in the position to calculate the firm's expected total annual logistics cost K as the sum of transport costs (Equation (4b)), ordering costs (Equation (5)), inventory holding costs (equations (7) and (13)) and stock-out costs (Equation (14)):

$$K = K_T + K_O + K_{MI} + K_{SI} + K_{SO}$$

After inserting each of these equations, rewriting and using Equation (A.1) to insert for $E(Q,R)$ and $B(Q,R)$ in Equation (14) if LTD is normally distributed (and A.4 if it is gamma distributed), the expected annual logistics costs can be written as:

$$K = [\gamma_1 + \psi(R)]\frac{x}{Q} + \gamma_2 Qx + \gamma_3 QY + \gamma_4 Q + \gamma_5 x + \gamma_6 Y + pH(R - \mu_L)$$

where

$$\gamma_1 = 2k_0(a_1 + \delta a_3) + (w + i_0)[2(\bar{t}_1 + \delta\bar{t}_3) + (u_1 + \delta u_3)] + b$$
$$\gamma_2 = pJ\eta^{-1}(1 + \delta)t_l$$
$$\gamma_3 = i_1 t_l$$
$$\gamma_4 = \tfrac{1}{2}pH(1 + \varepsilon)$$
$$\gamma_5 = (w + i_0 + w_l)t_l + P + pJ\eta^{-1}(\bar{t}_1 + \delta(\bar{t}_2 + \bar{t}_3))$$
$$\gamma_6 = 2k_1(a_1 + \delta a_3) + i_1[2(\bar{t}_1 + \delta\bar{t}_3) + (u_1 + \delta u_3)]$$
$$\psi(R) = \begin{cases} \pi\alpha(R) + (pH + \hat{\pi})x^{-1}\beta(R) & \text{if LTD is Normal} \\ \pi(\alpha\beta(1 - G_1(R)) - R(1 - G_0(R))) & \text{if LTD is Gamma} \end{cases} \quad (15)$$

Definitions of $\alpha(R)$ and $\beta(R)$ can be found in Appendix A. Except for transport costs and inventory holding costs during transport, K is a standard expression of average annual logistics cost in the tradition of Hadley and Whitin.

4.1. Cost minimisation

The decision maker's problem is to minimise total logistics cost subject to $C_{\min} \leq C \leq C_{\max}$ and $Y \geq x$. Since $C = QY/x$ (from Equation (3)), this may be written as

$$\begin{array}{lcl} Max_{Q,Y,R} & - & K \\ \text{s.t.} -QY & \leq & -C_{\min}x \\ QY & \leq & C_{\max}x \\ -Y & \leq & -x \end{array} \quad (16)$$

The constraints in (16) imply that Q and Y will be strictly positive, but no restriction is placed on R. Problem (16) is a non-linear mathematical programming problem (Sydsæter and Hammond, 2012). It can be solved numerically, but since it is a three-dimensional optimisation problem with non-linearities, it is computationally burdensome if the number of firms is large. In Appendix B we show how the Kuhn-Tucker conditions can be derived and used to simplify the problem. The solution itself – the optimal (Q^*,Y^*,R^*) and the logistics cost function K^* – is presented in Table 1 in the form of four cases, one and only one of which applies for each particular set of parameters.

In all four cases R^* will be the solution to the following equation, where definitions of the functions $\alpha(R)$, $g_1(R)$, $g_0(R)$ and $G_0(R)$ are provided in Appendix A (Equations A.2 - A.4 and the corresponding text):

Table 1
The four potential solutions: all variables as a function of R only.

Case 1	$Q^* = \sqrt{\frac{x(\gamma_1+\psi(R)+\gamma_6 C_{\min})}{\gamma_2 x+\gamma_4}}$	$Y^* = C_{\min} x\sqrt{\frac{\gamma_2 x+\gamma_4}{x(\gamma_1+\psi(R)+\gamma_6 C_{\min})}}$	R^* is the solution to (17)
	$K^* = 2\sqrt{x(\gamma_1+\psi(R^*)+\gamma_6 C_{\min})(\gamma_2 x+\gamma_4)} + (\gamma_5+\gamma_3 C_{\min})x + pH(R^*-\mu_L)$		
Case 2	$Q^* = C_{\min}$	$Y^* = x$	R^* is the solution to (17)
	$K^* = (\gamma_1+\psi(R^*))xC_{\min}^{-1} + ((\gamma_2+\gamma_3)x+\gamma_4)C_{\min} + (\gamma_5+\gamma_6)x + pH(R^*-\mu_L)$		
Case 3	$Q^* = \sqrt{\frac{(\gamma_1+\psi(R))x}{(\gamma_2+\gamma_3)x+\gamma_4}}$	$Y^* = x$	R^* is the solution to (17)
	$K^* = 2\sqrt{(\gamma_1+\psi(R^*))x((\gamma_2+\gamma_3)x+\gamma_4)} + (\gamma_5+\gamma_6)x + pH(R^*-\mu_L)$		
Case 4	$Q^* = C_{\max}$	$Y^* = x$	R^* is the solution to (17)
	$K^* = (\gamma_1+\psi(R^*))xC_{\max}^{-1} + ((\gamma_2+\gamma_3)x+\gamma_4)C_{\max} + (\gamma_5+\gamma_6)x + pH(R^*-\mu_L)$		

$$pHQ^* = \begin{cases} \pi x\left(1-F\left(\frac{R-\mu_L}{\sigma_L}\right)\right) + (pH+\hat{\pi})\alpha(R) & \text{if LTD is normal} \\ \pi x(\alpha\beta g_1(R) + 1 - Rg_0(R) - G_0(R)) & \text{if LTD is gamma} \end{cases} \quad (17)$$

Equation (17) is the first-order condition for minimising K with respect to the reorder point R, as shown in Appendix A.

By inserting Q^* from Table 1, Equation (17) becomes one equation in one variable. Thus, the non-linear three-dimensional minimisation problem (16) have been reduced to solving Equation (17) in R for each of the four cases, inserting the solution back in the table to obtain values of Q^*, Y^* and K^*. The true solution will be the case with the lowest K^*, given that the constraints still hold.

Appendix B shows that the logistics cost function K^* is in fact continuous in annual freight volume x not only inside each case, but also at the borders between cases. So as x increases, K^* shifts smoothly from Case 1 to 2, 3 and 4. There are economies of scale everywhere, but not equally strong over the whole range of cases. In Fig. 2 in the next section, an example of the transition from Case 1 to Case 3 as x increases is shown for a set of numerical values. The top right panel displays the change in Q as x increases; $Q < C_{\min}$ corresponds to Case 1; $Q = C_{\min}$ corresponds to Case 2; and $Q > C_{\min}$ corresponds to Case 3 ($C_{\min} = 0.901$). The top left panel displays the corresponding changes in each of the cost elements. As can be seen from the figure, the policy variables as well as the cost components are smooth functions of x, also at the boundaries between cases.

Appendix B also provides a characterization of each case in the form of relationships between the parameters (including R) that must apply if this case contains the solution.

4.2. Properties of the solution

We remark that in Case 1 and 3, shipment size is increasing in transport distance and transport time, γ_1. In Case 1, it is increasing in marginal vehicle operating costs γ_6 as well. Thus the unrealistic feature of the ordinary EOQ model pointed out in Section 5.5 of Combes (2009), namely that transport distance does not influence optimal shipment size, does not exist here. Taxes and fees that influence the kilometre costs k and the vehicle capital costs i will affect the optimal solution in much the same way as transport distance and transport time itself. The same applies to labour taxes.

The variables that do *not* influence the shipment size in any case are those belonging to γ_5, including the long-haul freight rate (by assumption), capital costs of the inventory-on-wheels during transport, and most of the time dependent costs during loading and unloading. The remaining costs, the stationary inventory costs γ_4, the inventory costs during loading and unloading γ_2, and the probably less important marginal time-dependent vehicle loading cost γ_3, influence the optimal shipment size in Case 1 and 3. The larger these costs are, the smaller the optimal shipment size.

The expressions for K^* in the four cases indicate that there are economies of scale in all cases, although more marked in cases 1 and 3 than in cases 2 and 4, since in the latter the shipment sizes are constant, and more market in Case 1 than in Case 3, since increasing the shipment size in Case 1 implies a higher vehicle utilisation rate. This result contradicts hypothesis 6.8 in Combes (2009), which says that there are no economics of scale in road freight transport.

5. A test case

The purpose of this test case is two-fold. First, to illustrate the responsiveness of the firm's optimal (Q,Y,R) policy in a realistic setting. Second, to show that these models can be calibrated with available data. The main dataset we use is the Norwegian Foreign Trade Statistics, which is unique in that it contains both the weight and the value of singular shipments, as well as the mode of transport. Furthermore, the commodity type, the ID of the domestic firm and the zip code of both the origin and the destination is recorded. Using this data, we construct a sample of consumer goods shipped from the Netherlands to Norway, and construct a test case in this market with median values for transport distances, transport times, unit prices and annual demand. For this test case we consider a transport chain consisting of the three legs road-sea-road, in which own transport is used in the distribution stages.

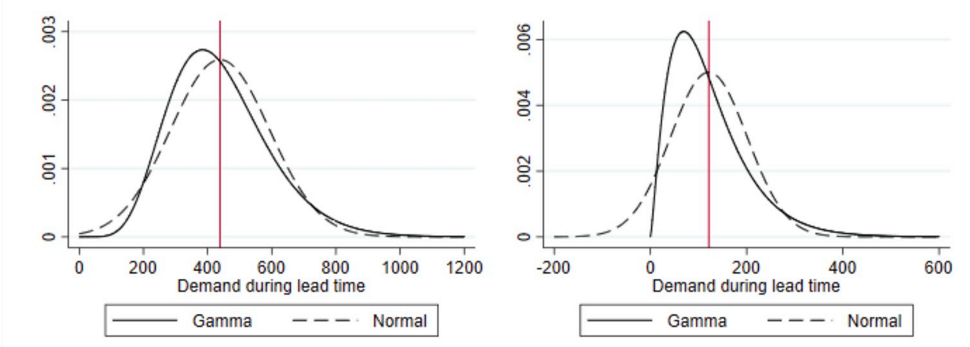
5.1. Data

The median firm-to-firm relationship in the transport of consumer goods between Norway and the Netherlands has an annual flow of 7.5 tonnes, with a FOB unit price of 45,000 NOK per tonne.[2] The median transport distance by road for this chain ($a_1 + a_3$) is 90 km, and the corresponding transport time by road is 1.5 h. Transport time by sea, including time in terminals, is found to be 96 h. For a full overview of the cost and uncertainty parameter values used and a description of how they are obtained, see Appendix C. There are no available data to inform us on the nature of the stock-out costs. Therefore, we display the results of the model for different stock-out cost levels. As will be shown, these parameters only have a minor impact on the results.

We present results of the case study both for normal and gamma distributed LTD. Using (8) and (9), we calculate these distributions, displayed in the lefthand panel of Fig. 1 where the vertical line indicates μ_L. For comparison, the distribution for the median door-to-door transport by road between Norway and Netherlands ($\bar{t} = 27$hours) is displayed in the right-hand panel.

The next section will show that the distributional assumption for LTD has minor impact on both policy variables and the minimised annual logistics costs (remember that we consider the left hand panel and not the right in this test case). The main reason for this is the low standard deviation of lead time compared to the expectation, leading to a higher shape parameter of the gamma distribution than what is conventionally used in constructed examples (e.g. Namit and Chen,

[2] On March 4th 2019, 1 NOK = 0,103 Euro.



$\mu_L = 438.7, \sigma_L = 154.1, \alpha = 8.1, \beta = 54.1$ $\mu_L = 121.5, \sigma_L = 79.7, \alpha = 2.3, \beta = 52.3$

Fig. 1. Distribution of lead time demand. Left: road-sea-road. Right: road. Unit: kg.

1999). Note that with our parameter values shorter lead times (e.g. domestic transport) will have more skewed gamma distributions, since in those cases the standard deviation of demand per time unit will dominate, leading to a higher coefficient of variation.

5.2. Solution

With the algebraic expressions from Section 4 (Table 1 and Equation (17)), minimising logistics cost is reduced to a smooth one-dimensional minimisation problem in R. The problem is easily solved with any software that employs non-linear optimisation methods, e.g. Excel or MatLab. The resulting policy variables and logistics costs are presented in Table 2. Y^* is not displayed, since for all cases $Q^* \in [C_{min}, C_{max}]$, meaning that no trucks have spare capacity and $Y^* = x$.

The differences between the six solutions in Table 2 are minor. This illustrates that generally, the effect of LTD is small for these parameter values – the sum of stock-out costs and inventory costs at the reorder point R^* constitutes a small share of annual logistics costs. Differences are as expected; costs increase in π and when changing the LTD distribution from normal to gamma, which has a longer right tail. As the cost per stock-out increases, the firm adapts by adjusting R^* appropriately such that stock-out costs are always close to zero while the stationary inventory cost is increasing. This suggests that the cost of additional inventory to counteract stock-outs is relatively low compared to the stock-out cost itself, given the parameter values used.

5.3. Sensitivity

One of the main issues with operationalising this framework is uncertainty regarding the true parameter values (or distribution of values). As demonstrated above, most values can be inferred or estimated from readily available firm and vehicle level micro data. The only parameter for which there is no information in available registers is the stock-out cost. Therefore, it is reassuring to see that model results, in particular K^* and Q^*, are robust for a wide range of stock-out cost parameter values. K^* is important in economic analyses since it covers all costs to the firm, while Q^* and C^* are needed to determine annual vehicle kilometres and vehicle size, which in turn determines transport externalities such as GHG emissions and congestion. The rest of this section will therefore present sensitivity analyses of the remaining main parameter values based on the model from the second column of Table 2, i.e. normally distributed LTD and stock-out cost equal to the commodity value.

Fig. 2 displays the results. Each row corresponds to changing one specific parameter of the model, displayed in the x-axis. The first column of graphs shows how key cost components at the optimum point change with parameter values, while the second column displays the corresponding optimal policy of the firm. Costs are measured in NOK/tonne, while the policy variables are measured in tonnes. The first row is the effect of varying annual demand and is therefore an illustration of returns to scale in firm size. As the demand per time unit increases, the firm responds by ordering larger shipments, thereby utilising returns to scale in transport costs to a larger extent. When $Q^* < C_{min}$ (=0.901) the firm is forced to pay for unutilised transport capacity, leading to an exponential increase in costs.

The second row displays the effect of changing the distance by road (the expected lead time has been changed correspondingly based on the average speed limit, i.e. by 1 h for every 60 km). The returns to scale in road transport costs becomes more prominent as the time and distance increases, leading to a larger shipment size. Had the shipment size not changed, the transport costs would have increased linearly in distance and the stationary inventory costs had been almost constant. The firm's optimal policy of increasing the shipment size illustrates the trade-off between higher transport costs and higher stationary inventory costs, which both increase at a diminishing rate.

The third row displays the effect of changing the standard deviation of the lead time demand, measured relative to the expected lead time (the baseline value is 0.35, corresponding to $\sigma_L = 154$ kg, and a value of 2 corresponds to $\sigma_L = 877$ kg). The right panel illustrates that variation of LTD mainly affects the reorder point, as expected, and the left panel illustrates how the firm's optimal policy translates into a trade-off between stationary inventory costs and stock-out costs.

Finally, the last row displays the effect of changing the costs per year and dollar of stationary inventory, H. This also translates into a trade-off between stationary inventory costs and transport costs and can be seen as a counterpart to the second row. The firm's optimal policy is to adjust its inventory by reducing Q^* as H increases, meaning that some returns to scale in transport costs is sacrificed.

This section illustrates two important points regarding the model. First, it seems to be capable of showing the room of manoeuvre that a firm facing uncertain lead time demand may have in adapting to new situations. Second, it reflects the sensitivity of transport costs per tonne to shipment size. The first of these aspects of the model is generally not present in freight models, while the second is generally not present in EOQ models.

6. Concluding remarks

The logistics cost model formulated here brings together two strands of literature. Depending on the viewpoint, it does so by bringing transport costs into an inventory model or by bringing inventory costs into a transport model. Also, it recognises the importance of the distribution phase of all forms of freight transport except door-to-door by truck. It does not identify shipment size with vehicle size, but assumes a minimal and a maximal vehicle size, so that optimal shipments can be smaller than, equal to or larger than the smallest vehicle, and smaller than the largest. It takes full account of uncertain demand and lead time and its implications for stockouts and safety stocks. The derived cost

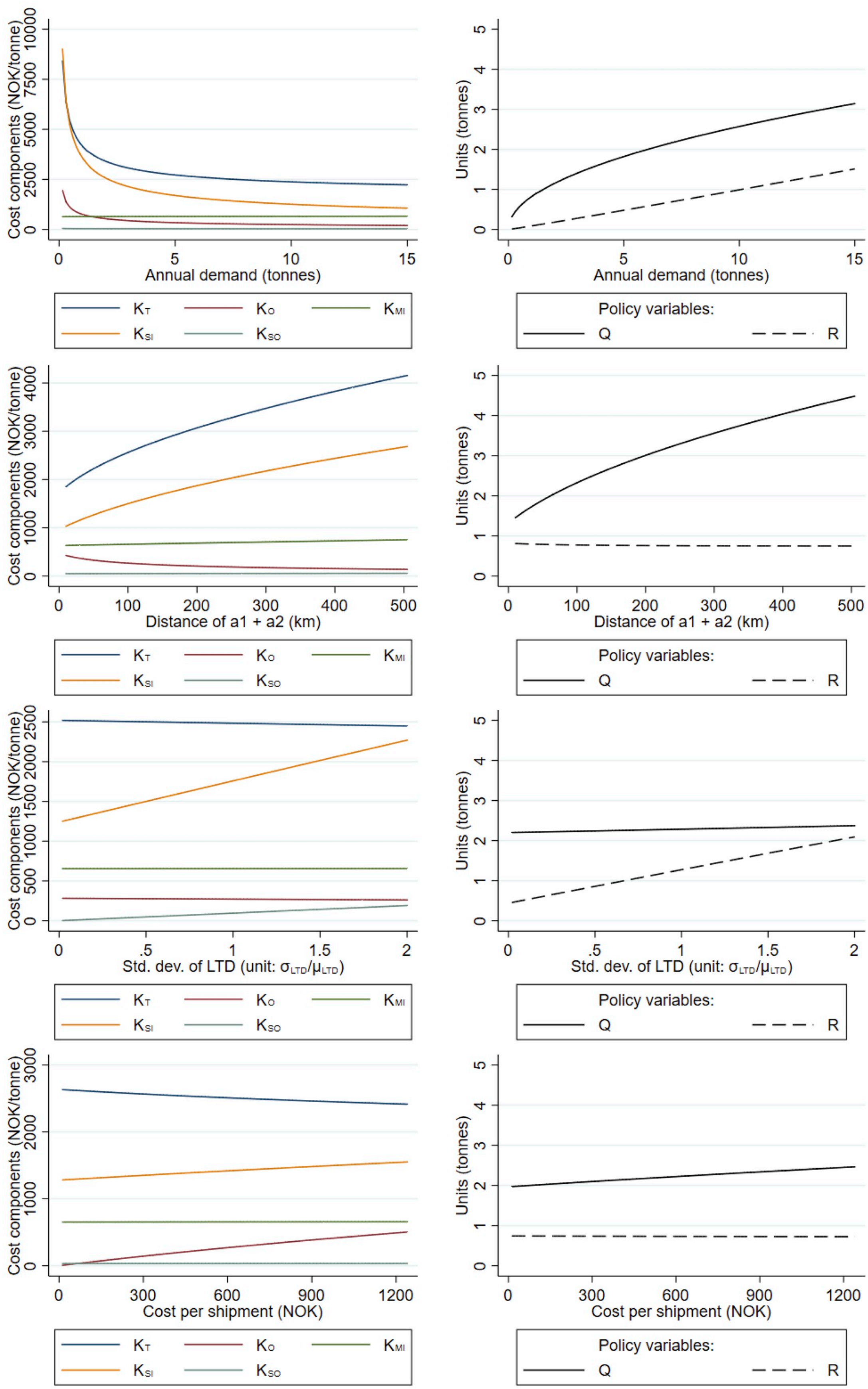


Fig. 2. Sensitivity analysis. Cost components and policy variables as functions of parameter values.

Table 2
Policy variables and cost elements for the transport chain road-sea-road. (1 NOK = 0,103 Euro).

π (% of p)	Normal lead time demand			Gamma lead time demand		
	50%	100%	500%	50%	100%	500%
Solution:	Case 3	Case 3	Case 3	Case 3	Case 3	Case 3
Policy variables (tonnes)						
Q^*	2.23	2.23	2.22	2.25	2.25	2.24
R^*	0.68	0.73	0.83	0.71	0.78	0.93
Cost elements (NOK/year)						
Transport	18,786	18,796	18,812	18,736	18,744	18,756
Ordering	2,083	2,086	2,091	2,069	2,071	2,075
Mobile inventory	4,911	4,911	4,911	4,912	4,912	4,912
Stationary inventory	10,473	10,669	11,056	10,635	10,922	11,534
Stock-out	277	250	208	409	388	356
Total	36,530	36,713	37,078	36,761	37,037	37,633

minimisation problem is analytically solvable in all variables except the reorder point, reducing the problem to a well-behaved optimisation problem in one dimension only. Tests in Excel with real life data solves the model for a single firm model in less than ¼ seconds on a standard desktop computer, producing reasonable results.

To our knowledge, no existing freight transport model incorporates inventory costs under uncertain transport time and/or demand. It is hoped that the model presented here can provide a suitable framework for future work to fill this gap. To make the framework operational, data on the variables entering the formulas must be had. It is thought that this can be done by collecting and processing micro-data at the level of the goods flow of firms, as illustrated in Section 5 and the Appendix C. Questions should concern the annual costs of the firm, not just single shipments. The most important points for future data collection would be to get better information about (1) stock-out costs, (2) demand distributions for (segments of) firms, and (3) (mode specific) transport time distributions. Data on (2) and (3) is needed to create compound lead time demand distributions.

The explicit representation of transport costs makes the model well suited for studies of the economic efficiency of transport prices and subsidies, economies of scale and scope etc. in specific cases. Since the framework can be calibrated and solved at the firm level, it may also be used to study how policies will affect different firms differently.

The chosen approach may also be used in cost benefit analyses for the appraisal of benefits to freight transport. It produces benefits to shippers of any changes in transport distances, times, prices and subsidies in the form of the difference between total logistics cost K in the do nothing case and the case of a new policy measure. (The carriers are always supposed to earn zero profit.) Benefits and costs to government and the environment will have to be computed by using separate accounts for the shipper, the government and the environment, and treat taxes and transfers as transfers between these accounts. While the private costs of the shippers are obtained directly from K, the real social costs are obtained by summing over all accounts.

Finally, incorporating mode choice is straightforward by comparing the annual logistics costs associated to each chain. Mode choice can be deterministic, such that firms choose the transport chain with the lowest annual cost given a transport chain-specific optimal policy. Alternatively, let annual chain-specific costs enter the right-hand side of a discrete choice model to be estimated. The potential for consolidation on all legs can be handled either exogenously (by applying parameters that reflect the cost after consolidation) or endogenously (by letting line haul cost parameters depend explicitly on aggregate transport demand).

Declaration of interest

None.

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Appendix A. Implications of distributional assumptions for LTD

To solve the cost minimisation problem of the firm, expressions for $E = E(Q, R)$ and $B = B(Q, R)$ are required. However, these will depend on the functional form of the statistical distribution of LTD. In this appendix, expressions for E and B are described for normal distributed LTD and gamma distributed LTD, respectively. These expressions are used in chapter 4 when formulating the logistics cost minimisation problem for normal and gamma distributed LTD.

Normal distributed LTD

Hadley and Whitin (1963, pp. 193-194) show, with a minor and mostly inconsequential simplification, that if LTD is normally distributed, then

$$E = E(Q, R) = x \cdot Q^{-1}\alpha(R)$$
$$B = B(Q, R) = Q^{-1}\beta(R) \tag{A.1}$$

where

$$\alpha(R) = \sigma_L\left[f\left(\frac{R-\mu_L}{\sigma_L}\right) - \frac{R-\mu_L}{\sigma_L}\left(1 - F\left(\frac{R-\mu_L}{\sigma_L}\right)\right)\right]$$
$$\beta(R) = \frac{1}{2}\sigma_L^2\left[\left(1 + \left(\frac{R-\mu_L}{\sigma_L}\right)^2\right)\left(1 - F\left(\frac{R-\mu_L}{\sigma_L}\right)\right) - \frac{R-\mu_L}{\sigma_L}f\left(\frac{R-\mu_L}{\sigma_L}\right)\right] \tag{A.2}$$

In (A.2), *f*(.) is the density function of a normalised normally distributed variable, and *F*(.) is its cumulative density function. To solve Equation (KT3) in Appendix B (to obtain Equation (17) in Section 4.1), derivatives of $\alpha(R)$ and $\beta(R)$ are needed. And since $F'(x) = f(x)$ and $f'(x) = -xf(x)$,

$$\alpha'(R) = (-1)\left(1 - F\left(\frac{R-\mu_L}{\sigma_L}\right)\right)$$
$$\beta'(R) = -\alpha(R) \tag{A.3}$$

Both functions are monotonously decreasing, and as both approach 0 as R goes to infinity, they are everywhere positive.

Gamma distributed LTD

To treat the gamma distributed case in a straightforward manner, we must assume that $B(Q,R) = 0$, which is only realistic if the impact on the logistics cost minimisation is small enough to be neglected. Note that $B(Q,R)$ affects the logistics costs through two channels – the stock-out costs and the stationary inventory costs. Thus, setting $B(Q,R)$ to zero is a realistic assumption if stock-out costs occur per unit instead of accumulating over time, so that $\hat{\pi} \approx 0$, and the contribution of $B(Q,R)$ to the stationary inventory cost is small. Note also that if the probability that more than $Q + R$ units are demanded in the duration of the lead time is close to zero, $B(Q,R)$ will be close to zero as well (since backorders immediately will be delivered when the first shipment arrives), meaning that the former requirements will hold (Hadley and Whitin, 1963, pg. 194). While we cannot test whether $\hat{\pi} \approx 0$ is a realistic assumption with our data, the other assumptions hold for all sensible parameter values.

The same assumptions are conventionally made in articles assuming gamma distributed lead time demand, see e.g. Namit and Chen (1999), Tyworth et al. (1996) or Tyworth and Ganeshan (2000). Under the restriction that $R > \mu_L$ and the assumption that B can be neglected, Tyworth et al. (1996) show that assuming gamma distributed LTD implies that

$$E = E(Q, R) = xQ^{-1}(\alpha\beta(1 - G_1(R)) - R(1 - G_0(R)))$$
$$B = 0 \tag{A.4}$$

where G_1 is the cumulative distribution function of Gamma $(\alpha + 1, \beta)$, and G_0 is shorthand for Gamma(α, β). Note that α and β here means the shape and scale parameter of the gamma distribution of lead time demand respectively, while $\alpha(R)$ and $\beta(R)$ refer to the expressions from Equation (A.2). The restriction $R > \mu_L$ is only relevant for the optimal policy when it is binding, and that has not been the case in any of the numerical examples implied by our cost parameters.

Finally, the parameters of the gamma distribution can be determined as follows (Tyworth and Ganeshan, 2000): $\alpha = \sigma_L^2/\mu_L^2$ and $\beta = \sigma_L^2/\mu_L$, where μ_L and σ_L can be obtained from (8) and (9). To solve Equation (KT3) in Appendix B (to obtain Equation (17) in Section 4.1), note that $G'(R) = g(R)$, i.e. the derivative of the cumulative distribution function is the probability density function.

Appendix B. Derivation of the logistics cost function

Problem (16) of the main text is a non-linear mathematical programming problem. To solve it, we form the Lagrangian L,

$$L = -K - \lambda_1(C_{\min}x - QY) - \lambda_2(QY - C_{\max}x) - \lambda_3(x - Y)$$

λ_1, λ_2 and λ_3 are called Lagrangian multipliers. Assuming the optimal Q and Y to be strictly positive, while no restriction is placed on R, the Kuhn-Tucker conditions for a maximum are

$$\frac{\partial L}{\partial Q} = \frac{\partial L}{\partial Y} = \frac{\partial L}{\partial R} = 0$$
$$\lambda_1 \geq 0 \;\; (=0 \;\text{ if }\; QY > C_{\min}x)$$
$$\lambda_2 \geq 0 \;\; (=0 \;\text{ if }\; QY < C_{\max}x)$$
$$\lambda_3 \geq 0 \;\; (=0 \;\text{ if }\; x < Y) \tag{B.1}$$

From $\frac{\partial L}{\partial Q} = \frac{\partial L}{\partial Y} = \frac{\partial L}{\partial R} = 0$ we derive the Kuhn-Tucker conditions for a maximum:

$$(\gamma_1 + \psi(R))xQ^{-2} - (\gamma_2 x + \gamma_3 Y + \gamma_4) + (\lambda_1 - \lambda_2)Y = 0 \tag{KT1}$$

$$-\gamma_3 Q - \gamma_6 + (\lambda_1 - \lambda_2)Q + \lambda_3 = 0 \tag{KT2}$$

$$pHQ = -\frac{\partial\psi(R)}{\partial R}x \tag{KT3}$$

Using Equation (A.3) from Appendix A for the normal distribution case, and the fact that in the gamma case, $G'(X) = g(x)$, i.e. the probability density function, the lefthand side of (KT3) can be written explicitly as

$$pHQ = \begin{cases} \pi x\left(1 - F\left(\frac{R-\mu_L}{\sigma_L}\right)\right) + (pH + \hat{\pi})\alpha(R) & \text{if LTD is normal} \\ \pi x(\alpha\beta g_1(R) + 1 - Rg_0(R) - G_0(R)) & \text{if LTD is gamma} \end{cases} \tag{B.2}$$

Equation (B.2) is of course identical with Equation (17) of the main text.

Since the three constraints can be binding or non-binding, there are 8 cases to consider. However, if all three constraints or only the first and second constraint is binding, there is an immediate contradiction unless $C_{\min} = C_{\max}$. Also, if none or only the second is binding, meaning that the corresponding Lagrangian multipliers are zero, there is contradiction in Kuhn-Tucker condition (KT2). We are left with four cases:

Case 1. *Only the first constraint is binding.* Here, $\lambda_2 = \lambda_3 = 0$ and $QY = C_{\min}x$. Solving (KT2) for λ_1 and using this and $Y = C_{\min}xQ^{-1}$ to eliminate λ_1 and Y in (KT1), we get

$$Q^* = \sqrt{\frac{x(\gamma_1 + \psi(R) + \gamma_6 C_{\min})}{\gamma_2 x + \gamma_4}} \tag{B.3}$$

$$Y^* = C_{\min}x\sqrt{\frac{\gamma_2 x + \gamma_4}{x(\gamma_1 + \psi(R) + \gamma_6 C_{\min})}} \tag{B.4}$$

Since $\psi(R)$ is a function of R, this is not an explicit solution. The optimal (Q^*,R^*) is found either by choosing a starting value for Q, and iterating between updating the R of (KT3) and Q until convergence, or by using any non-linear optimisation algorithm (e.g. Newton's method) to find the R that minimises K, where (B.3) and (B.4) has been inserted for Q and Y. Because the third constraint is not binding, we have $Y > x$. By (B.4), then, it follows that in this case,

$$\sqrt{\frac{x(\gamma_1 + \psi(R) + \gamma_6 C_{\min})}{\gamma_2 x + \gamma_4}} < C_{\min} \tag{B.5}$$

which means that $Q^* < C_{min}$. The smallest possible truck is used, but it will not be filled to capacity.

Case 2. *The first and third constraints are binding.* $\lambda_2 = 0$. From the two binding constraints, we immediately get

$$Q^* = C_{\min} \tag{B.6}$$

$$Y^* = x \tag{B.7}$$

The optimal Q^* is larger in this case than in the first case, as the smallest possible vehicle is used, but with a full truck-load. R^* follows from (KT3). Using these optimal values in (KT1) and (KT2), we get explicit expressions for λ_1 from (KT1), and using this optimal λ_1, we may find λ_3 from (KT2). The necessary condition (B.8) for Case 2 to apply consists of two inequalities. One of them follows from $\lambda_1 \geq 0$, and the other follows from $\lambda_3 \geq 0$ (See B.1).

$$\sqrt{\frac{(\gamma_1 + \psi(R))x}{(\gamma_2 + \gamma_3)x + \gamma_4}} \leq C_{\min} \leq \sqrt{\frac{x(\gamma_1 + \psi(R) + \gamma_6 C_{\min})}{\gamma_2 x + \gamma_4}} \tag{B.8}$$

Case 3. *Only the third constraint is binding.* From the binding constraint we immediately have

$$Y^* = x \tag{B.9}$$

It also follows that $\lambda_1 = \lambda_2 = 0$, and therefore by KT1,

$$Q^* = \sqrt{\frac{(\gamma_1 + \psi(R))x}{(\gamma_2 + \gamma_3)x + \gamma_4}} \tag{B.10}$$

(B.10) is not an explicit solution, but (Q^*, R^*) may be found from (B.10), (B.9) and (KT3) by the same procedure as in Case 1. From the non-binding constraints, however, we may deduce that

$$C_{\min} < Q^* < C_{\max} \tag{B.11}$$

Case 4. *The second and third constraints are binding.* Here, $\lambda_1 = 0$ and from the binding constraints,

$$Q^* = C_{\max} \tag{B.12}$$

$$Y^* = x \tag{B.13}$$

From the condition that $\gamma_2 \geq 0$ it follows by KT1 that

$$\sqrt{\frac{(\gamma_1 + \psi(R))x}{(\gamma_2 + \gamma_3)x + \gamma_4}} \geq C_{\max} \tag{B.14}$$

Between them, the four cases contain the optimal solution to problem (16) of the main text. The four necessary conditions for each case to apply – (B.5), (B.8), (B.11) and (B.14) – may be used to indicate in advance which one of the four cases contains the solution, or to control the solution afterwards. As will be seen by studying the inequality signs, the four cases make for a smooth transition from case to case of the policy variables Q^*, Y^*, R^* as parameters change.

The Bolzano-Weierstrass theorem says that a continuous function on a non-empty compact set achieves a maximum and a minimum there. The function K to be minimised is continuous in the policy variables. The existence of a solution to the problem is therefore guaranteed by the Bolzano-Weierstrass theorem provided we can assume that there is a strictly positive minimal order quantity, because in that case, the feasible region is compact. So one of the four cases gives us the solution.

As annual volume x or for instance the costs per shipment $\gamma_1 + \psi(R^*)$ increase, the solution moves from the first of our four cases to the second, the third and the fourth. The logistic cost function K^* as a function of annual volume x and the prices is continuous inside the cases and at the borders between cases, i.e., everywhere.

Appendix C. Data

All parameter values used in the numerical example in Section 4 are displayed in table C1. After the table follows a description of how the various parameter values were obtained.

Table C1
Parameter values used in the case study. (1 NOK = 0,103 Euro)

Model parameter	Symbol	Value	Units
Transport/inventory cost parameters			
Expected annual demand	x	7.5	tonnes
Price/commodity value	p	45,000	NOK/tonne
Transport distance, road	$a_1 + a_3$	90	km
Expected transport time, road	$\bar{t}_1 + \bar{t}_3$	1.5	hours
Unproductive time, road	$u_1 + u_3$	0.74	hours
Expected time, sea transport and ports	$\bar{t}_2$	96	hours
Loading/unloading time, road	t_l	0.34	hours/tonne
Distance dependent cost component	k_0	3.26	NOK/km
Distance dependent cost component	k_1	0.159	NOK/tonne-km
Time dependent cost component	$w + i_0$	417.8	NOK/hour
Time dependent cost component	i_1	1.57	NOK/tonne-hour
Loading/unloading costs, road	w_l	1083	NOK/hour
Fixed price of second leg	P	976	NOK/tonne
Non-transport ordering cost	b	620	NOK/shipment
Stationary inventory cost	H	0.09	NOK/year-value
Mobile inventory cost	J	0.43	NOK/year-value
Unit cost per backorder (varies by case)	π	45,000	NOK/tonne
Cost per year of a backorder	$\hat{\pi}$	0	NOK/tonne
Minimum vehicle size	C_{min}	0.901	tonnes
Maximum vehicle size	C_{max}	13.52	tonnes
Business hours per year	η	1667	hours
Derived parameters of lead time demand			
Expected lead time	μ_T	97.5	hours
Standard deviation of lead time	σ_T	8.13	hours
Expected demand per business hour	μ_D	4.50	kg/hour
Standard deviation of demand	σ_D	15.16	kg/hour
Expected lead time demand	μ_L	438.66	kg
Standard deviation of lead time demand	σ_L	154.11	kg
Gamma dist. shape parameter	α	8.10	kg
Gamma dist. scale parameter	β	54.14	kg

From The Norwegian Foreign Trade Statistics we observe that the median firm-to-firm relationship in the transport of consumer goods between Norway and the Netherlands has an annual demand of 7.5 tonnes, with a FOB unit price of 45,000 NOK per tonne. The annual demand is defined as the sum of shipments per domestic firm-ID – foreign zip code – commodity type combination. This will be a correct representation of a foreign firm unless the same domestic firm exports/imports the same commodity type (on a 6 digit level) to more than one foreign firm in the same zip code.

The median transport distance by road for this chain ($a_1 + a_3$) is 90 km, and the corresponding transport time by road is 1.5 h. To obtain these values we have employed a time-minimising route-finding algorithm (Dijkstra, 1959) on a road network from OpenStreetMap (Ramm et al., 2011) between the coordinates of ports and the centroids of zip-codes observed in the data. The transport times are based on the registered speed limit, including obligatory breaks.

As time and distance dependent cost parameters by road we use the fitted line from regressions of time- and distance-based costs on transport capacity for the set of freight vehicles used in the National Freight Transport Model of Norway, NGM. For time and distance costs, see Grønland (2015, table 3.3). The distance dependent costs are maintenance costs, fuel costs and tire replacement costs, while time cost components are wage, capital costs, annual registration fee, insurance and administration costs. The vehicle capacity in weight will depend on the voluminousity of the commodity. Here we focus on the broad commodity group "consumer goods", for which the various truck capacities can be found in Madslien et al. (2015, pg. 88). $[C_{min}, C_{max}]$ are defined as the capacity of the smallest and largest vehicle, respectively.

Transport time by sea (the median from time tables) and handling and waiting time in terminals (obtained from the NGM model, conditional on the transport frequency) is estimated to be 96 h in total. The unit cost of the sea leg is the average unit cost for consumer goods between the port of Oslo and Amsterdam/Rotterdam port as predicted from NGM. As stationary and mobile inventory cost and loading and unloading time and cost parameters we use standard values from NGM (Madslien et al., 2015).

The uncertainty parameters are obtained from Hovi et al. (2018), and more thoroughly documented there. The standard deviations of demand and lead time are obtained from separate sources as percentages of expected demand per business hour and expected transport time, respectively. In future work it would be useful to estimate these distributions as well, and not only their first and second moments.

The Norwegian Commodity Flow Survey (CFS) is used to quantify the standard deviation of demand per hour to 337% of its expected value. Here, approximately 60 million shipments are reported with information on product type, origin, destination, quantity (weight) and time when the shipment is ordered (measured in minutes).

The standard deviation of lead time for sea transport is calculated to be 8.2% of the expected transport time by sea. To obtain this, The National Coastal Administration's AIS-data and data on port calls from port administration systems are used. Containing call date and time, arrival and departure as well as an identification number for each ship, these data makes it possible to follow every ship observed. However, since lead time is not observed, time of calls is used instead.

The standard deviation of transport time by truck to and from terminals is extracted from Google Maps over several weeks at given times. Nine typical links from terminals in Norway are considered and the standard deviation is found to be 17% of the transport time. When deriving the standard deviation of total lead time, it is assumed that the standard deviations of truck and sea lead time are independent, and that the standard deviation of terminal time is zero.

References

Allen, W.B., Liu, D., 1993. An inventory-transport model with uncertain loss and damage. Transport. Res. Part B 19 (5), 447–456.

Bagchi, U., Hayya, J.C., Chu, C.-H., 1986. The effect of lead time variability: the case of independent demand. J. Oper. Manag. 6 (2), 159–177.

Baumol, W.J., Vinod, H.D., 1970. An inventory theoretic model of freight transport demand. Manag. Sci. 16 (7), 413–421.

Burgin, T.A., 1975. The gamma distribution and inventory control. Oper. Res. Q. 26 (3), 507–525.

Cobb, B.R., 2013. Mixture distributions for modelling demand during lead time. J. Oper. Res. Soc. 64 (2), 217–228.

Cobb, B.R., Johnson, A.W., Rumí, R., Salmerón, A., 2013. Inventory management with log-normal demand per unit time. Comput. Oper. Res. 40 (2013), 1842–1851.

Cobb, B.R., Johnson, A.W., Rumí, R., Salmerón, A., 2015. Accurate lead time demand modeling and optimal inventory policies in continuous review systems. Int. J. Prod. Econ. 163, 124–136.

Combes, F., 2009. The Choice of Shipment Size in Freight Transport. Ph.D dissertation. Economies et finances, Université Paris-Est.

Constable, G.K., Whybark, D.C., 1978. The interaction of transportation and inventory decisions. Decis. Sci. J. 9 (4), 688–699.

Das, C., 1974. Choice of transport service: an inventory-theoretic approach. Logist. Transp. Rev. 10 (2), 181–187.

Das, C., 1975. Effect of lead time on inventory: a static analysis. Oper. Res. Q. 26 (2), 273–282.

Das, C., 1976. Approximate solution to (Q,R) for gamma lead time demand. Manag. Sci. 22 (9), 1043–1047.

Das, C., 1983. Inventory control for lognormal demand. Comput. Oper. Res. 10 (3), 267–276.

Dijkstra, E.W., 1959. A note on two problems in connexion with graphs. Numer. Math. 1 (1), 269–271.

Grønland, S.E., 2015. "Cost Models for Freight Transport and Logistics – Base Year 2012. Institute of Transport Economics, Oslo, Norway TØI Report 1435/2015.

Grubbström, R.W., 1995. Modelling production opportunities – an historical overview. Int. J. Prod. Econ. 41 (1–3), 1–14.

Hadley, G., Whitin, T.M., 1963. Analysis of Inventory Systems. Prentice-Hall Inc., Englewood Cliffs, N.J.

Hovi, I.B., Jordbakke, G., Pinchasik, D., Killi, M., Auråen, E., 2018. Variability in Transport Time and Demand for Freight Transport in Norway. Institute of Transport Economics, Oslo, Norway TØI Report 1644/2018, (In Norwegian.).

Jansson, J.O., 1984. Transport System Optimization and Pricing. Wiley, Chichester.

Langley Jr., C.J., 1981. The inclusion of transportation costs in inventory models: some considerations. J. Bus. Logist. 2 (1), 106–125.

Larson, P.D., 1988. The economic transport quantity. Transport. J. 28 (2), 43–48.

Larson, P.D., 1989. The integration of inventory and quality decisions in logistics: an analytical approach. J. Bus. Logist. 10 (2), 106–122.

Lloret-Batlle, R., Combes, F., 2012. Estimation of an inventory-theoretic model of mode choice in freight transport. Transport. Res. Rec. 2378, 13–21.

Madslien, A., Steinsland, C., Grønland, S.E., 2015. The National Norwegian Freight Model. How to Use the Model. Institute of Transport Economics, Oslo, Norway TØI Report 1429/2015.

Namit, K., Chen, J., 1999. Solutions to the < Q,R > inventory model for gamma lead-time demand. Int. J. Phys. Distrib. Logist. Manag. 29 (2), 138–154.

Ramm, F., Topf, J., Chilton, S., 2011. OpenStreetMap: Using and Enhancing the Free Map of the World. UIT Cambridge, Cambridge.

Sydsæter, K., Hammond, P., 2012. Essential Mathematics for Economic Analysis, fourth ed. Pearson Education Limited, Harlow, England.

Tyworth, J.E., 1991. The inventory-theoretic approach in transportation selection models: a critical review. Logist. Transp. Rev. 27 (4), 299–318.

Tyworth, J.E., 1992. Modeling transportation-inventory trade-offs in a stochastic setting. J. Bus. Logist. 13 (2), 97–124.

Tyworth, J.E., 2018. A note on lead-time paradoxes and a tale of competing prescriptions. Transport. Res. Part E 109 (2018), 139–150.

Tyworth, J.E., Ganeshan, R., 2000. A note on solutions to the (Q, r) inventory model for gamma lead-time demand. Int. J. Phys. Distrib. Logist. Manag. 30 (6), 534–539.

Tyworth, J.E., Guo, Y., Ganeshan, R., 1996. Inventory control under gamma demand and random lead time. J. Bus. Logist. 17 (1), 291–304.

Tyworth, J.E., L. O'Neil, L., 1997. Robustness of the normal approximation of lead-time demand in a distribution setting. Nav. Res. Logist. 44 (2), 165–186.

Tyworth, J.E., Zeng, A.Z., 1998. Estimating the effect of carrier transit time performance on logistics cost and service. Transport. Res. Part A 32 (2), 89–97.

Vernimmen, B., Witlox, F., 2003. The inventory-theoretic approach to modal choice in freight transport: literature review and case study. Brussels Econ. Rev. 46 (2), 5–29.

Vernimmen, B., Dullaert, W., Willemé, P., Witlox, F., 2008. Using the inventory-theoretic framework to determine cost-minimizing supply strategies in a stochastic setting. Int. J. Prod. Econ. 115 (2008), 248–259.

Wanke, P., Leiva, V., 2015. Exploring the potential use of the Birnbaum-Saunders distribution in inventory management. Math. Probl. Eng. 2015 Article ID 827286. https://www.hindawi.com/journals/mpe/2015/827246/.